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Strain Based Cylindrical Rectangular Finite Element for Analysis of Cylindrical Shell under Uniformly Distributed Loads

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Abstract— A cylindrical rectangular finite element is developed in this paper. The element has six nodal degrees of freedom at each of the four corner nodes, (Three general external degrees of freedom and three rotations). The displacement fields of the element satisfy the exact requirements of rigid body modes of motion. Shallow shell formulation is used and the element is based on an independent strain assumption insofar as it is allowed by the compatibility equations. A cylindrical shell problem for which a previous solution exists is first analyzed using the new element to test the efficiency of the element . The element is then used in the analysis of cylindrical shell subjected to uniformly distributed load varying sinusoidal along its length in addition to symmetric sinusoidal edge loads present along its longitudinal boundaries. The distribution of various components of stresses is obtained and the effect of radius-length ratio is also presented to give the designer an insight for the behavior of such structures.

Keywods – Strain-based, cylindrical rectangular finite elements

I. INTRODUCTION

Considerable attention has been given applying the finite element method of analysis to curved structures. The early work on the subject was presented by Grafton and Strome (1963) who developed conical segments for the analysis of shells of revolution. Jones and Strome (1966) modified the method and used curved meridional elements which were found to lead to considerably better results for the stresses.

Further research led to the development of curved rectangular as well as cylindrical shell elements (Connor et al, 1967; Bogner et al, 1967; Cantin et al, 1968 and Sabir et al, 1972). However, to model a shell of arbitrary or triangular shape by the finite element method, a triangular shell element is needed. Thus many authors have been occupied with the development of curved triangular shell elements and consequently many elements (Lindberg et al, 1970 and Dawe, 1975) resulting in an improvement of the accuracy of the results. However, this improvement is achieved at the expense of more computer time as well as storage to assemble the overall structure matrix. Meanwhile, at the United Kingdom, a simple alternative approach has been used to the development of curved elements.

This approach is based on determining the exact terms representing all the rigid body modes together with the displacement functions representing the straining of the element by assuming independent strain functions insofar as it is allowed by the compatibility equations. This approach has successfully employed in the development of curved shell elements (Ashwell et al, 1971, 1972; Sabir et al, 1975, 1982, 1983, 1987; El-Erris, 1994, 1995 and Mousa, 1994, 1998, 2012,2015). These elements were found to yield faster convergence with other available finite elements.

The strain-based approach is employed in the present paper to develop a new rectangular strain-based cylindrical element having six degrees of freedom at each corner node. The new element is first tested by applying it to the analysis of a clamped barrel vault for which a previous solution exists. The work is then extended to the analysis of a cylindrical shell subjected to uniformly distributed load varying sinusoidal along its length and symmetric sinusoidal edge loads present along its longitudinal boundaries. The distribution of the various components of stresses is obtained to give designers an insight into the behaviour of such structures.

II. THEORETICAL CONSIDERATION OF THE DISPLACEMENT FUNCTIONS FOR THE NEW CYLINDRICAL ELEMENT

A rectangular shallow cylindrical shell element and the associated curvilinear coordinates are shown in Figure 1.

For the shown system of curvilinear coordinates, the simplified strain displacement relationship for the cylindrical shell elements can be written as:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{w}{r}\ ,\ \varepsilon_y = \frac{\partial v}{\partial y},\ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$k_x = \frac{\partial^2 w}{\partial x^2}\ ,\ k_y = \frac{\partial^2 w}{\partial x^2},\ k_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} \qquad (1)$$

Where u, v and w are the displacements in the x, y and z axes; $\varepsilon_x, \varepsilon_y$ and ε_{xy} are the in-plane direct and shearing strains; k_x, k_y, and k_{xy} are the changes in-direct and twisting curvatures and r is the principal radius of curvature.

The above six components of strain can be considered independent, as they are a function of the three displacements u, v and w and must satisfy three additional compatibility

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equations. These compatibility equations are derived by eliminating u, v and w from Equation 1, hence, they are:

$$\partial^2\varepsilon_x/\partial y^2+\partial^2\varepsilon_y/\partial x^2-\partial^2\varepsilon_{xy}/\partial x\partial y+k_y/r=0$$
$$\partial^2 k_{xy}/\partial x-2\partial k_x/\partial y=0 \qquad (2)$$
$$\partial k_{xy}/\partial y-2\partial ky/\partial x=0$$

In order to keep the element as simple as possible and to avoid the difficulties associated with internal non-geometric degrees of freedom, the developed element should possess six degrees of freedom at each of the four corner nodes: u, v, w, θ_x, θ_y and ϕ. Thus, the shape functions for a rectangular element should contain twenty four independent constants.

To obtain the displacement fields due to rigid body movements, all the six strains given by Equation 1 are equated to zero and the resulting partial differential equations are integrated to yield:

$$u=-a_1\frac{x}{r}-a_2\left(-\frac{x^2}{2r}\right)+a_3\frac{xy}{r}+a_4+a_6y$$
$$v=-a_1\frac{y}{r}-a_3\left(\frac{x^2}{2r}\right)+a_5-a_6x \qquad (3)$$
$$w=-a_1+a_2x+a_3y$$

These displacement fields are due to the six components of the rigid body displacements and are represented in terms of the constants a_1 to a_6. If the element has six degrees of freedom for each of the four corner nodes, the displacement fields should be represented by twenty four independent constants. Having used six constants for the representation of the rigid body modes, the remaining eighteen constants are available for expressing the displacements due to the strains within the element. These constants can be apportioned among the strains in several ways. For the present element, the following way is proposed:

$$\varepsilon_x=a_7+a_8y+a_{21}y^2+2a_{22}xy^3$$
$$-\frac{1}{r}\left[a_{16}\frac{y^2}{2}+a_{17}\frac{xy^2}{2}+a_{18}\frac{y^3}{6}+a_{19}\frac{xy^3}{6}\right]$$
$$\varepsilon_y=a_9+a_{10}x-a_{21}y^2-2a_{22}xy^3$$
$$\varepsilon_{xy}=a_{11}+2a_{22}x+2a_{24}xy \qquad (4)$$
$$k_x=a_{12}+a_{13}x+a_{14}y+a_{15}xy$$
$$k_y=a_{16}+a_{17}x+a_{18}y+a_{19}xy$$
$$k_{xy}=a_{20}+[2a_{14}x+a_{15}x^2+2a_{17}y+a_{19}y^2]$$

Equation 4 is derived by first assuming the un-bracketed terms and adding the terms between brackets to satisfy the compatibility condition (Equation 2). It is then equated to the corresponding expressions in terms of u, v and w from Equation 1 and the resulting equations are integrated to obtain:

$$u=a_7x+a_8xy-a_{10}\frac{y^2}{2}+a_{11}\frac{y}{2}+a_{21}xy^2+a_{22}x^2y^3$$
$$+a_{24}y^2+a_{12}\frac{x^3}{6r}+a_{13}\frac{x^4}{24r}+a_{14}\frac{x^3y}{6r}+a_{15}\frac{x^4y}{24r}$$
$$+a_{20}\left(\frac{x^2y}{4r}\right)$$
$$v=-a_8\frac{x^2}{2}+a_9y+a_{10}xy+a_{11}\frac{x}{2}-a_{21}x^2y \qquad (5)$$
$$-a_{22}x^3y^2+a_{23}x^2-a_{14}\frac{x^4}{24r}-a_{15}\frac{x^5}{120r}$$
$$+a_{20}\left(-\frac{x^3}{12r}\right)$$
$$w=-a_{12}\frac{x^2}{2}-a_{13}\frac{x^3}{6}-a_{14}\frac{x^2y}{2}-a_{15}\frac{x^3y}{6}-a_{16}\frac{y^2}{2}$$
$$-a_{17}\frac{xy^2}{2}-a_{18}\frac{y^3}{6}-a_{19}\frac{xy^3}{6}-a_{20}\frac{xy}{2}$$

The complete displacement functions for the element are obtained by adding the corresponding expressions for u, v and w from Equations 3 and 5. The translational degrees of freedom for the element are u, v, w. The three rotations about the x, y and z axes are given by:

$$\theta_x=\frac{\partial w}{\partial y}=a_3-a_{14}\frac{x^2}{2}-a_{15}\frac{x^3}{6}-a_{16}y-a_{17}xy$$
$$-a_{18}\frac{y^2}{2}-a_{19}\frac{xy^2}{2}-a_{20}\frac{x}{2} \qquad (6)$$
$$\theta_y=-\frac{\partial w}{\partial x}=-a_2+a_{12}x+a_{13}\frac{x^2}{2}+a_{14}xy+a_{15}\frac{x^2y}{2}$$
$$+a_{17}\frac{y^2}{2}+a_{19}\frac{y^3}{6}+a_{20}\frac{y}{2}$$
$$\phi=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=a_6-a_8x+a_{10}y-2a_{21}xy-3a_{22}x^2y^2$$
$$-a_{23}x-a_{24}y-a_{14}\frac{x^3}{6r}-a_{15}\frac{x^4}{24r}-a_{20}\left(\frac{x^2}{4r}\right)$$

The Stiffness matrix [*K*] for the shell element is then calculated in the usual manner, i.e.,

$$[K]=[C^{-1}]^T\left\{\iiint[B^T]DBdv\right\}[C^{-1}] \qquad (7)$$

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where B and D are the strain and elasticity matrices, respectively, and C is the matrix relating the nodal displacements to the constant a_1 to a_{24}. B can be calculated from eqns (1) , (3) , (4) and D is given by substituting the matrices B and D into eqn. (5). The integration within the bracketed terms of eqn (5) are carried out explicitly and the rest are computed to obtain the stiffness matrix [K] .

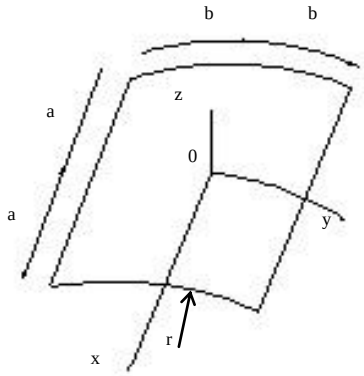


Fig. 1. Coordinate system for a rectangular cylindrical shell element

III. Patch Convergence Test

This test is to be considered which is frequently used to test the performance of the shell elements is that of Scordelis-Lo Roof having the geometry as shown in figure (2). The shell has the following dimensions and material properties: thickness, t= 0.03 m, r = 3m, L = 6m, $\alpha = 40^{\circ}$, modulus of elasticity, E= 3 Pa, Poisson's ratio, μ= 0.0, density, ρ = 0.625 Pa. The straight edges are free while the curved edges are supported on rigid diagrams along their plan considering the symmetry of the problem only one quarter of the roof is analyzed.

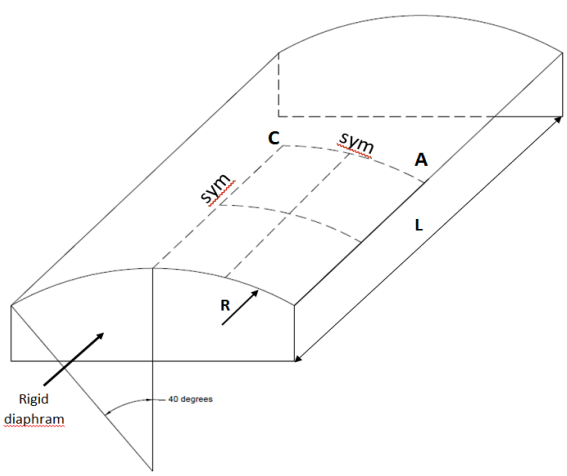
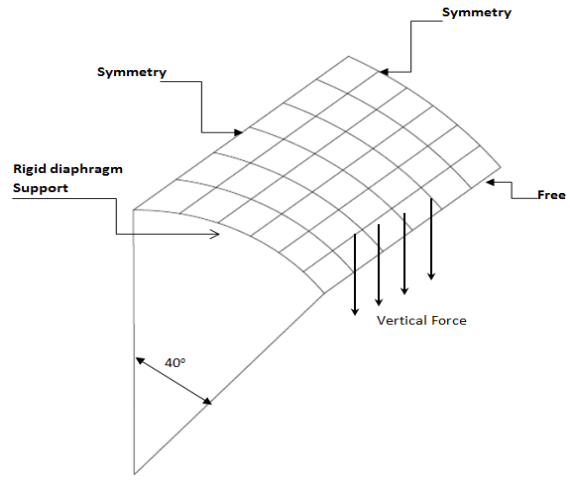


Fig. 2. Geometry and finite element mesh of the roof



The results obtained by the new present element for the vertical displacement at the midpoint B of the free edge and the center C of the roof are compared to other kinds of shells elements (Batoz *et al*, 1992 and Hamadi *et al*, 2000). The analytical solution of this problem is based on the shallow shell theory is given by Scordelis and Lo (1969). Convergence curves (Fig 3 and Fig 4) show that the convergence of the present element faster convergence than the other. Then the new present element would be more efficient to use it in the analysis of proposed cylindrical shell under loads.

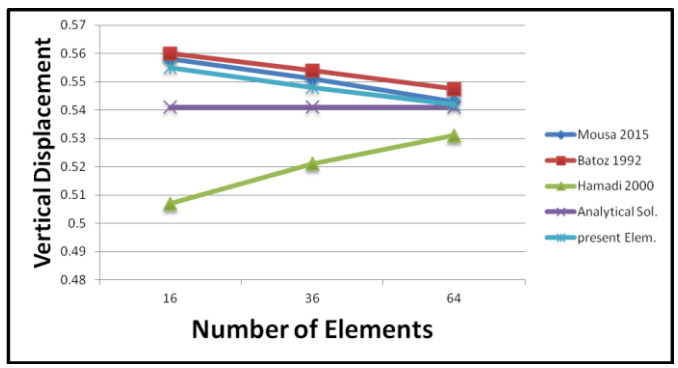


Fig. 3 Convergence curve for the deflection, w, at point C

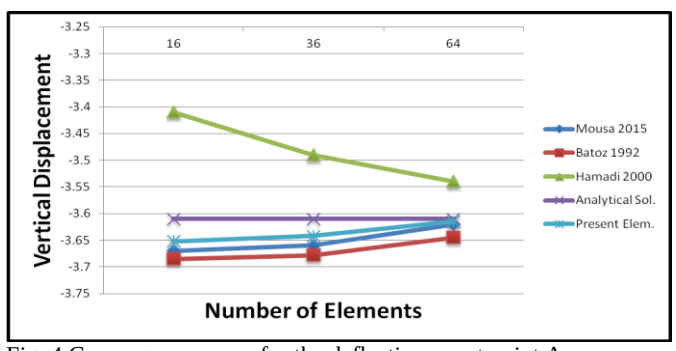


Fig. 4 Convergence curve for the deflection, w, at point A

IV. Problem Considered

An open barrel cylindrical shell is analyzed under uniform surface load varying sinusoidally in addition to the sinusoidal symmetric edge loads present along the longitudinal

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boundaries. The geometric properties of the shell is shown in figure 5. The shell has the following geometry and properties; L=26.67 m, R=8 m, $\alpha = 45^{\circ}$, thickness t=80 mm, E=25$x10^{6}\frac{kN}{m^2}$, $\mu = 0.2$ and the applied load q = 3.25 $\frac{kN}{m^2}$, the distribution of this load is based on the following equation: $q \sin\left(\frac{\pi x}{L}\right)$

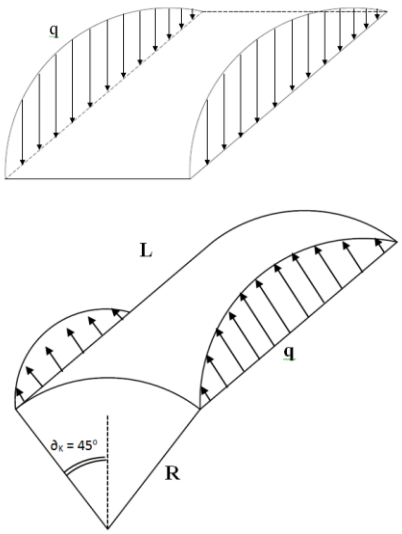

Fig. 5. Geometry and loads on the shell

The figure (6-10) show the stress resultant for the problem considered. It is seen that the obtained results on the basis of the proposed present element very closely agree with these obtained from the analysis based on classic flexure theory procedure discussed by Chandrasekaran *et al* (2009).

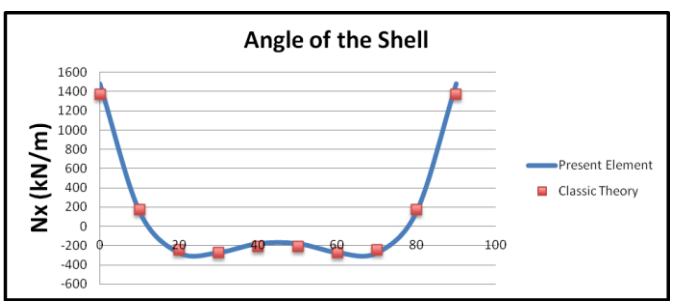

Fig. 6 Axial stress resultant Nx, at distance L/2

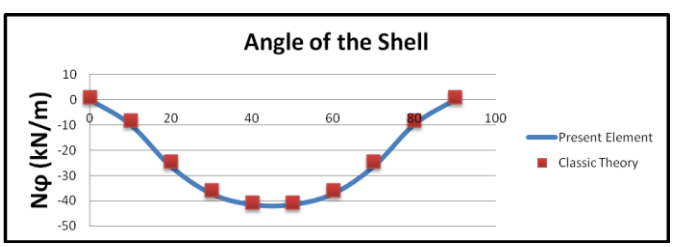

Fig. 7 Axial stress resultant Nϕ, at distance L/2

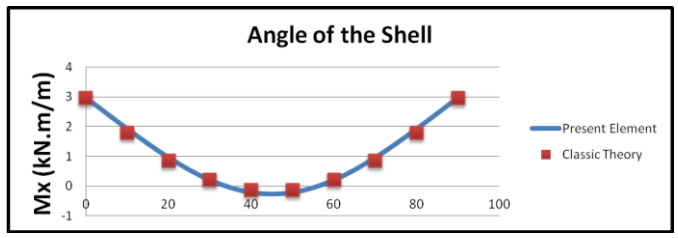

Fig. 8 Bending stress resultant Mx, at distance L/2

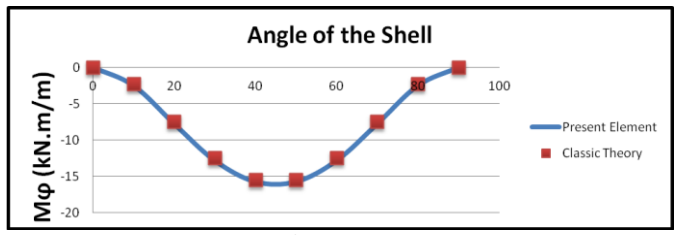

Fig. 9 Bending stress resultant Mϕ, at distance L/2

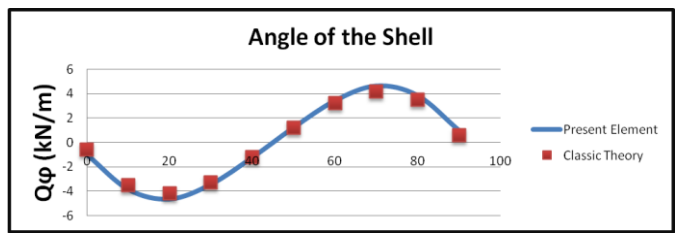

Fig. 10 Stress resultant Qϕ, at distance L/2

The work is extended to study the effect of radius length ratio on behavior of the cylindrical shell problem. The figures (11-15) show the stresses resultant for different radius length ratio (R/L = 0.3, 0.4, 0.5, 0.6). It's clear that the various stress resultants are decreased by increasing the radius length ratio.

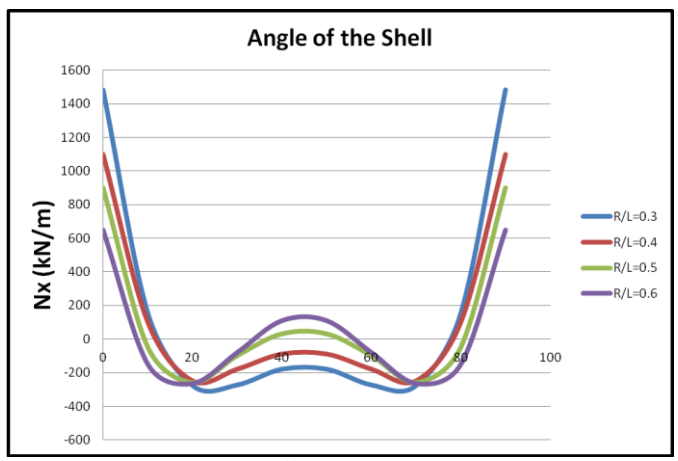

Fig. 11 Stress resultant Nx, at different radius length ratio

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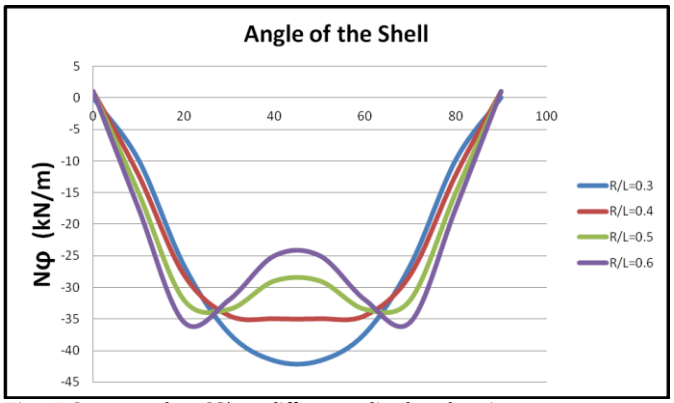


Fig. 12 Stress resultant Nϕ, *at different radius length ratio*

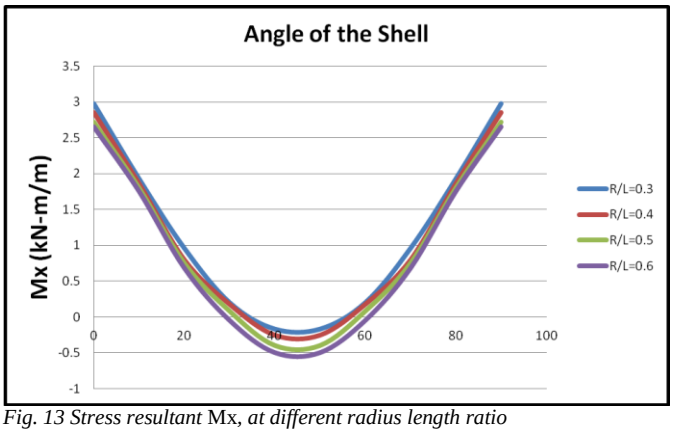


Fig. 13 Stress resultant Mx, *at different radius length ratio*

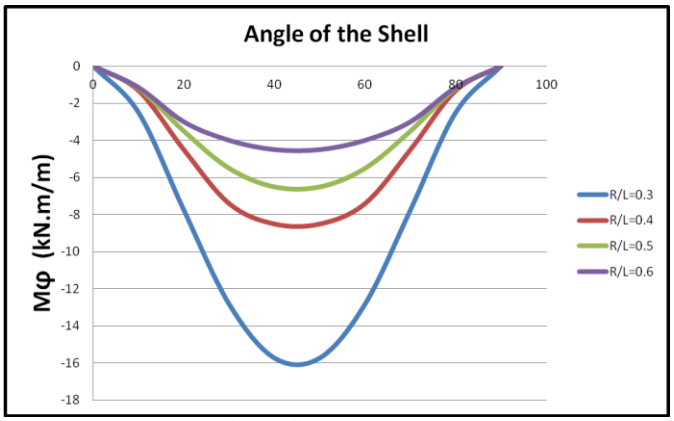


Fig. 14 Stress resultant Mϕ, *at different radius length ratio*

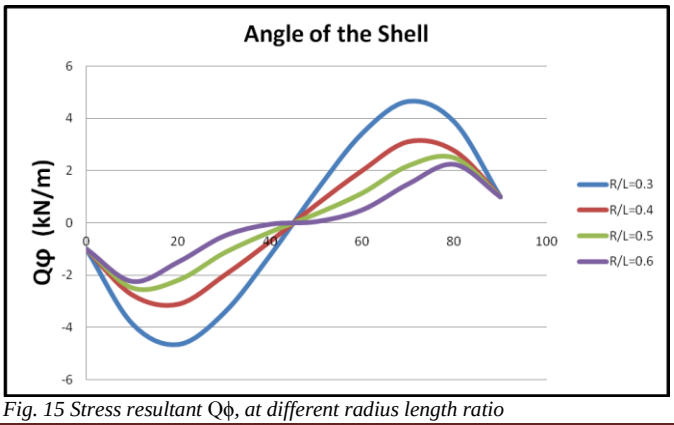
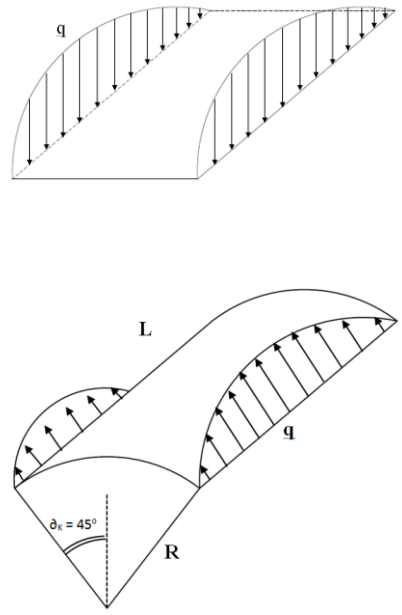
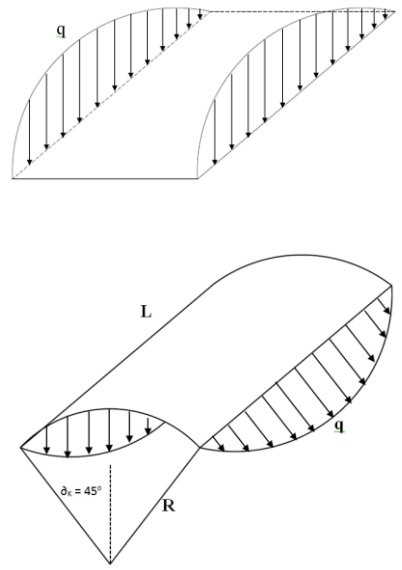


Fig. 15 Stress resultant Qϕ, *at different radius length ratio*

More investigation is carried out to show the behaviour of the cylindrical shell
subjected to uniformly distributed load varying sinusoidally along its length in addition to different symmetric edge loads present along its longitudinal boundaries .Two cases are considered , Case 1 which represent the surface load plus to shear edge load ,and case 2 represents the surface load plus to axial edge load as shown in figure 16.



Case 1: Surface load + Shear edge load



Case 2. Surface load +Axial edge load

Fig.16. shell under uniform distributed load

The figures (17-21) show the stress resultants in the problem for both cases .It is clear that a minor change in axial stress resultants , *Nx* , *Nϕ* ,while a

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considerable change is shown at the cylinder edge in Mx and $Q\phi$ and at the crown of the cylinder in $M\phi$.

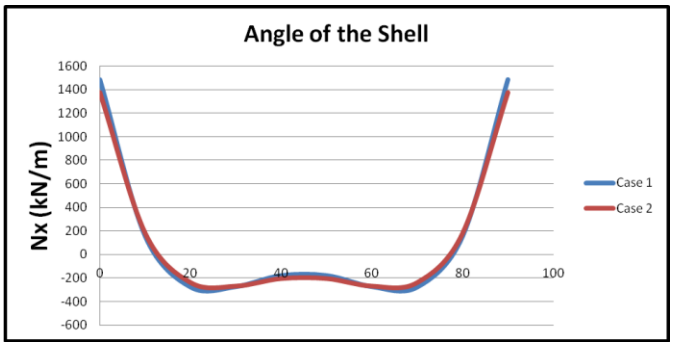

Fig.17. Axial stress resultant Nx, at distance L/2

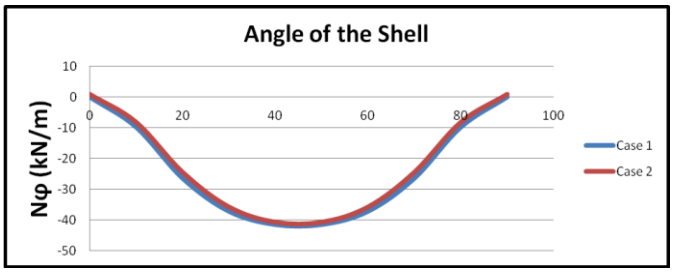

Fig. 18. Axial stress resultant Nϕ, at distance L/2

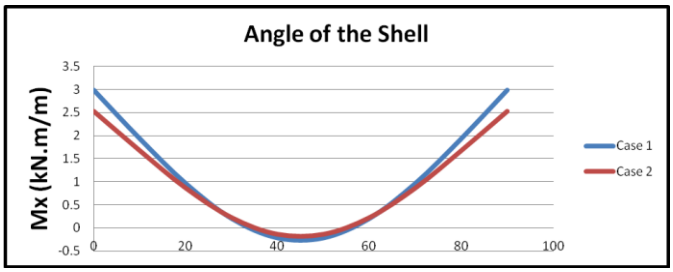

Fig.19. Bending stress resultant Mx, at distance L/2

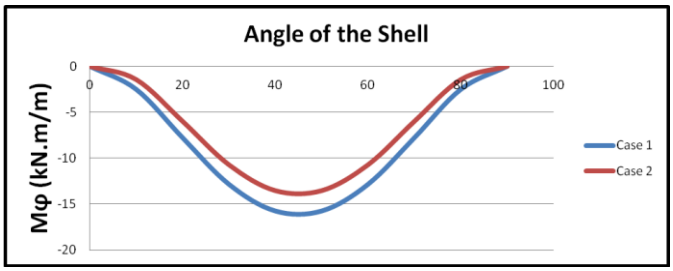

Fig. 20. Bending stress resultant Mϕ, at distance L/2

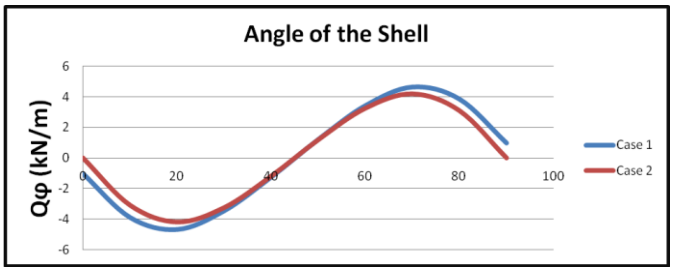

Fig. 21. Stress resultant Qϕ, at distance L/2

V. Conclusion

A new rectangular strain-based finite cylindrical element is developed using shallow shell formulation. The element has the six degrees of freedom at each corner node.

The developed rectangular cylinder finite element is first applied to analysis of a clamped barrel vault. The results for the deflections are presented and show that the developed element has a very good agreement results. This element is then used to analyze a cylindrical shell subjected to uniformly distributed load varying sinusoidal along its length and symmetric cylindrical edge sinusoidal load presented along its longitudinal boundaries.

The effect of the radius length ratio on the various stresses components is presented.

It's seen the various stresses components are decreased by increasing the radius length ratio. This gives the designers an insight for the behavior of such structures.

References

[i] Grafton P.E. and Strome. D.R., (1963), "Analysis of axisymmetric shells by the direct stiffness method" *AIAA. J.* 1, 2342-2347.

[ii] Jones. RE and Strome. d.R., (1966), "Direct stiffness method analysis of shells of revolution utilizing curved dements", *AIAA. J.*, (4), 15159-1525.

[iii] Conner. J.J. and Brcbbia, C., (1967), "Stiffness matrix for shallow rectangular shell element". *J. Eng. Mech. Div. ASCE*, 93. No EMS, 41-65.

[iv] Bogner. F K. Fox. R L. and Schmit. I. A., (1967), "A cylindrical shell discrete element", *J. AIAA.*, 5(4), 745-750.

[v] Cantin. G and Clough. R. W., (1968), "A curved cylindrical shell finite element" *AIAA J.* 16, 1057-1062.

[vi] Sabir. A.B. and Lock A.C., (1972), "A curved cylindrical shell finite element" *Int. J. Mech. Sci.* 14, 125-135.,

[vii] Lindbergy. G.M. Cowper. G. R. and Olson. M.D., (1970), "A shallow shell finite element of triangular shape" *Int. J. Solids struc.* 14, 1133-1156.

[viii]Dawe. D.J., (1975), "High order triangular finite element for shell and analysis" *Int. J. Solids struct.* 11, 1097-1110.

[ix] Ashwell. D. G. Sabir. A. B. and Roberts T. M., (1971), "Further studies in the application of curved finite element to circular studies" *Int. J. Mech. Sci.* 13. 507-517.

[x] Ashwell. D. G. and Sabir. A. B, (1972), "A new cylindrical shell finite element based on simple independent strain functions". *Int. J. Mech. Sci.* 14, 171-183.

[xi] Sabir. A. B., (1975), "Stiffness matrices for general deformation (out of plane and in-plane) of curved beam members based on independent strain functions", T*he Maths of Finite Elements and Applications II. Academic Press.* 34, 411-421.

[xii] Sabir. A. B. and Charchafchi. T. A., (1982), "Curved rectangular and general quadrilateral shell element for cylindrical shells" *The Maths of Finite Elements and Applications IV. Academic Press* 231-238.

[xiii]Sabir. A. B., (1983), "Strain based finite for the analysis of cylinders with holes and normally intersecting cylinders". *Nuclear Eng. and Design.* 76. 111-120.

[xiv] Sabir. A B. (1987), "Strain based shallow spherical shell element" *Proceedings Int. Conf. on the Mathematics of Finite Element and Applications*, Bronel University.

[xv] El-Erris, H. F., (1994), "Behavior of hipped roof structures", Proc. 2nd. MCE Eng. Con., CEI, Baghdad.

[xvi] El-Erris, H. F., (1995), "Effect of eccentricity of crown and edge beams on the behavior of hipped roof structures", accepted for publishing in the journal of military college of engineering, Baghdad.

[xvii] El-Erris, H. F., (1997), "A general shell finite element with in-plane rotation" J. Al-Muhandis of the University of Technology, Baghdad, No. 130.

[xviii] Mousa. A. I. and Sabir. A. B., (1994), "Finite element analysis of fluted conical shell roof structures". *Computational structural engineering in practice. Civil Comp. Press*, ISBN 0-948748-30-X pp 173-181.

[xix] Mousa. A. I., (1998), "Finite element analysis of a gable shell roof" *Advances in Civil and Structural of Engineering Computing for Paretic civil-comp press*, 26-268.

[xx] Mousa A.I, Aljuadi A.E, Kameshki E. Dahman, N, (2012), "New strain cylindrical rectangular finite element for the analysis of arch dam" Canadians Journal on Environment, Construction and Civil Engineering, Canada.

[xxi] Mousa , A, and Djoudi, M. , (2015), "A shallow shell finite element for linear and nonlinear of spherical shells " International journal of engineering and scientific ,IJENS. Vol.15, Issue 05 , pp 24-29.

[xxii] Mousa , A. and Djoudi M. , (2015), " New strain based triangular finite element for the vibration of circular cylindrical shell with oblique ends " . International journal of civil and environmental engineering , IJCEE . Vol.15 , Issue 05 , pp 6-11

[xxiii] J. L. Batoz and G. Dhatt, (1992), "Modelisation des structures par elements finis". Vol. 3: Coques, Eds Hemes, Paris.

[xxiv] Hamadi D. and M. T. Belarbi, (2000), "Experimental and numerical analysis of an elliptical paraboloid shell structure" The Eight Arab Structural Engineering Conference, 21-23, pp 109-118, Cairo University Faculty of Engineering.

[xxv] Scordelis C. and Lo, K. S., (1969), "Computer analysis of cylindrical shells", J. Amer. Concrete Institute, Vol. 61, pp 539-561.

[xxvi] Chandrasekaran et al, (2009), "Design aids for fixed support reinforced concrete cylindrical shells under uniformly distributed loads" International Journal of Engineering Science and Technology, Vol. 1, No. 1, pp 148-171.